APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Summer 2017 (May)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

This is a closed-book exam. No calculator is allowed. Start your answer on its corresponding question page. If you use extra pages, print your name and the question number clearly at the top of each extra page. Hand in all answer pages.

Date of Exam: May 24, 2017

Time: 9:00 AM – 1:00 PM

A1. Solve the initial value problem:

$$u_{tt} - u_{xx} = 0$$

$$u_t(x, 0) = 0, \qquad u_t(x, 0) = \begin{cases} \cos \pi(x-1) & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases},$$

Find and draw the solutions at $t = 1, 2, 3$, respectively for

(a). $x \in (-\infty, \infty)$.

(b). $x \in (-\infty, 3)$ with $u_x(3, t) = 0$.

(c). $x \in (0, \infty)$ with $u(0, t) = 0$.

(d). $x \in (0, 3)$ with $u(0, t) = u(3, t) = 0$.

(e). $x \in (0, 3)$ with $u_x(0, t) = u_x(3, t) = 0$.

(f). $x \in (0, 3)$ with $u_x(0, t) = u(3, t) = 0$.

A2. Given the conservation law

$$u_t + \frac{1}{2}\left(u^2\right)_x = 0$$

with the initial condition

$$u_0(x) = \begin{cases} 0 & x \leq 0 \\ 2 & 0 < x \leq 4 \\ 1 & 4 < x < 5 \\ 0 & x > 5 \end{cases}.$$

(a). Find the equation of characteristics, draw characteristics on the $x - t$ plane, mark the time when characteristics intercepts.

(b). Find the weak solution at $t = 2$.

(c). Find the Riemann solution with the initial condition

$$u(x, 0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}.$$

(d). Find the Riemann solution for $\frac{x}{t} = 0$ and $t > 0$.

A3. Given the boundary value problem of the Laplace equation,

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega : (0,1) \times (0,1).$$

$$\begin{aligned} u(x,0) &= x(1-x), \qquad u(x,1) = 2x(x-1) \\ u(0,y) &= 2y(1-y), \qquad u(1,y) = y(y-1). \end{aligned} \tag{1}$$

(a). Solve the problem through the separation of variable method.

(b). Find the maximum and minimum of the solution in the closure $\overline{\Omega}$.

A4. Determine the number of zeros (counting multiplicities) of the polynomial $z^7 - 4z^3 + z - 1$ which are situated inside the circle $|z| = 1$.

A5. Prove that an analytic function $f : \mathbf{C} \to \mathbf{C}$ satisfying $|f(z)| \leq 4$ for any $z \in \mathbf{C}$ must be constant.

B6. Given three node points: $a - h$, a, $a + h$, and the function values at these points as f_1, f_2, f_3.

(a). Find the Lagrangian polynomial interpolation.

(b). Find the best numerical approximation for $f'(x)$ and $f''(x)$ at $a - h$ and a using the given function values. What are the order of truncations errors?

(c). Find the integral of $\int_{a-h}^{a+h} f(x)dx$ using the Simpson's method, estimate the truncation error.

(d). If we know $f(x)$ in the entire interval of $[a - h, a + h]$, what is the highest order of numerical integration you can get if only two points are used? Elaborate your calculation.

B7. For the boundary value problem

$$y'' + p(x)y' + q(x)y = r(x),$$

with boundary condition

$$y(0) = a, \quad y(L) = b.$$

(a). Find a transformation $y \to w$ such that the boundary condition becomes homogeneous: $w(0) = w(L) = 0$. What is the new equation for w?

(b). Under what condition the boundary value problem for w has unique solution?

(c). Describe the shooting method to solve the problem, including the procedures and a psudo-code.

(d). In each iteration step of your shooting method, there is a truncation error which advances as

$$e_{n+1} = Ce_n^r$$

Estimate the order r of your shooting method.

B8. Prove that a conservative method for a scalar hyperbolic conservation law with a Lipschitz continuous numerical flux $F(U;j)$ is TV-stable if for each initial data u_0 there exist some $\Delta t_0, R > 0$ such that $TV(U^n) \leq R$ for any $n, \Delta t$ such that $\Delta t < \Delta t_0$ and $n\Delta t < t_{end}$.

B9. Using Taylor series, one finds that the Lax Wendroff and the Beam-Warming methods for the linear advection equation $v_t + av_x = 0$ are 3rd order accurate for the following modified PDE: $v_t + av_x = \mu v_{xxx}$, where the coefficient μ for each scheme is, respectively,

$$\mu_{LW} = \frac{\Delta x^2}{6} a \left(\frac{\Delta t^2}{\Delta x^2} a^2 - 1 \right), \quad \text{and} \quad \mu_{BW} = \frac{\Delta x^2}{6} a \left(2 - \frac{3\Delta t}{\Delta x} a + \frac{\Delta t^2}{\Delta x^2} a^2 \right).$$

(a) Derive the dispersion relation for each modified PDE and find the corresponding phase and group velocities.

(b) Describe qualitatively the behavior of solutions obtained with both schemes using the following step-function initial condition: $v(x, 0) = 1$ if $|x| \leq 1$, and $v(x, 0) = 0$ if $|x| > 1$, as $R = \frac{|a|\Delta t}{\Delta x}$ changes within the corresponding stability range ($R \leq 1$ for the Lax-Wendroff scheme and $R \leq 2$ for the Beam-Warming scheme). Select several typical values of R for each scheme that illustrate the nature of dispersive solutions.

B10. Using the Gerschgorin Circle Theorem, analyze stability of the FTCS scheme for the following IBVP

$$v_t = \nu v_{xx}, \quad x \in (0,1),\, t > 0$$

$$v(0,t) = 0; \quad v_x(1,t) + 5v(1,t) = \sin(5t), \quad t \geq 0$$

$$v(x,0) = f(x), \quad x \in [0,1],$$

when the discretization of the right boundary condition is performed using a symmetric, 2nd order finite difference approximation. Is the condition you obtained necessary, sufficient, or both? Justify your conclusion.